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The stability of a thermally radiating stratified shear layer, including self-absorption

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A linear stability analysis is applied to a stably stratified, thermally radiating shear layer. The grey Milne–Eddington approximation is employed as a radiation model. In contrast to a previously reported optically thin analysis, no inviscid instability exists, in the limit of vanishing horizontal wavenumber, for this self-absorbing model. The inviscid neutral-stability boundary (Richardson number *vs.* dimensionless wavenumber) for the Milne–Eddington approximation converges to the optically thick limit as the optical depth of the shear layer is increased. As the optical depth of the shear layer is decreased, the inviscid Milne–Eddington neutral-stability boundary approaches the optically thin limit, although not uniformly in the wavenumber. For fixed mean velocity gradient and fluid properties, the inviscid critical Richardson number approaches infinity as the optical depth of the shear layer approaches zero. Viscous effects neutralize this radiative destabilization, and the critical Richardson number eventually returns to zero as the optical depth continues to decrease. A shear-layer thickness exists for which the viscous critical Richardson number is a maximum. For shear depths greater than this thickness, self-absorption effects increase the stability; and for shear depths less than this thickness, viscous effects increase the stability. Results of the analysis are applied to the atmospheres of Venus and the earth. A critical Richardson number somewhat above the non-radiating value of $\frac{1}{4}$ (although below the previously reported optically thin value) is found for the lower troposphere of the earth. No substantial effect is found for the earth's lower stratosphere or for the 100 km level above Venus.

1. Introduction

In a recently reported linear stability analysis, Dudis (1973) examined the effect of thermal radiation in reducing the stability of a stably stratified shear layer. The analysis was assumed to be valid for shear layers with depths less than an appropriate photon mean-free-path length (taken to be the reciprocal of the Planck mean absorption coefficient). Under this condition temperature disturbances were assumed to be optically thin, radiative transfer was modelled by a linearized form of the Newtonian law of cooling and any self-absorption was neglected. Inviscid calculations resulted in a complete radiative destabilization of the shear layer; the Richardson number became infinite as the horizontal wavenumber of the disturbances approached zero. The inclusion of viscosity modified this behaviour, and finite critical Richardson numbers were found for all finite Reynolds numbers.

Application of these results to the atmospheres of the earth and Venus yielded critical Richardson numbers considerably greater than the non-radiating critical value of $\frac{1}{4}$. The calculations for Venus were made in an attempt to assess the possibility of radiative transfer being the mechanism which maintains relatively large turbulent mixing coefficients in the stably stratified upper Venusian atmosphere. These large mixing coefficients have been postulated in the literature as one way to explain the photochemical stability of upper atmospheric carbon dioxide (see Donahue 1971; Ingersoll & Leovy 1971). With this earlier optically thin analysis, plausible atmospheric parameters led to a critical Richardson number greater than the actual atmospheric value.

The effect of thermal radiation in suppressing turbulent fluctuations in a stably stratified shear flow was investigated by Townsend (1958). He considered the equations for the mean-square turbulent velocity and temperature fluctuations and, through assumptions regarding viscous dissipation, heat conductivity and disturbance correlation coefficients, was able to determine a relationship between the critical Richardson number and the radiative cooling rate. Goody (1964) applied Townsend's results to several regions in the earth's atmosphere. For the stratosphere he found essentially no radiative effects, and for the lower troposphere he calculated a doubling of the critical Richardson number for turbulent eddies smaller than a few metres. Results of the optically thin stability analysis (Dudis 1973) gave considerably larger increases in the critical Richardson numbers for both of these regions.

One of the possible difficulties with the optically thin stability analysis concerns the neglect of self-absorption effects. This neglect becomes increasingly questionable for small wavenumber (large wavelength) disturbances where the maximum radiative destabilization was found. The present paper will consider self-absorption in order to explore more carefully the small wavenumber optically thin instability. Hyperbolic-tangent mixing-layer profiles of mean velocity and potential temperature, investigated in the optically thin case, will be employed.

Radiation will be treated by employing the grey Milne–Eddington (or differential) approximation. First, we shall consider a shear layer whose depth is much greater than a photon mean-free-path length. Under this condition the Milne–Eddington approximation reduces to the optically thick approximation, and radiative transfer will be entirely diffusive. Inviscid computations will be performed; it will be shown that the viscous problem is equivalent to that considered by Miller & Gage (1972), concerning the effect of reduced Prandtl numbers on the critical Richardson number.

This analysis will be followed by a more general treatment of the Milne–Eddington approximation; we shall investigate the effect of varying the optical depth of the shear layer. In particular, we shall be concerned with comparing the results of this approximation, as the optical depth goes both to zero and infinity, with the results of the optically thin and thick approximations, respectively. The specific numerical examples for the earth and Venus considered by Dudis (1973) will be reconsidered, employing the results of the Milne–Eddington approximation.

2. Governing equations

The notation employed in this paper will be identical to that of Dudis (1973). We shall consider a stratified shear layer with velocity $\mathbf{U}(z) = (U(z), 0, 0)$ and potential temperature $\theta(z)$, where these profiles are given by

$$U = \Delta U \tanh(z/L), \quad \theta = T_0 + \Delta\theta \tanh(z/L).$$

The linearized Boussinesq equations and boundary conditions governing disturbances to this basic state are

$$\frac{d\mathbf{u}}{dt} + w\frac{\partial \mathbf{U}}{\partial z} = -\frac{1}{\rho_0}\nabla p - \frac{\mathbf{g}}{T_0}\phi + \nu\nabla^2\mathbf{u}, \tag{2.1a}$$

$$\frac{d\phi}{dt} + w\frac{\partial\theta}{\partial z} = \frac{\kappa}{\rho_0 c_p}\nabla^2\phi - \frac{\nabla.\mathbf{q}}{\rho_0 c_p}, \tag{2.1b}$$

$$\nabla.\mathbf{u} = 0, \tag{2.1c}$$

$$\mathbf{u}, \phi, p, \mathbf{q} \to 0 \quad \text{as} \quad z \to \pm\infty, \tag{2.1d}$$

where $d/dt = \partial/\partial t + U\,\partial/\partial x$. Once again, $\mathbf{u} = (u, v, w)$, ϕ, p and $\mathbf{q}$ represent velocity, temperature, pressure and radiative heat flux disturbances, respectively. The acceleration due to gravity $\mathbf{g} = (0, 0, -g)$, a reference density ρ_0, a reference temperature T_0, the kinematic viscosity ν, the coefficient of thermal conductivity κ and the specific heat at constant pressure c_p are all assumed constant.

Radiation will be modelled by the Milne–Eddington approximation (Vincenti & Kruger 1965, p. 492). This approximation assumes spectrally invariant absorption and an isotropic closure relation between directional moments of the radiation intensity to give a differential approximation to the equation of radiative transfer. In dimensional linearized form it is given by

$$\alpha^{-1}\nabla(\nabla.\mathbf{q}) - 3\alpha\mathbf{q} - 16\sigma T_0^3\nabla\phi = 0, \tag{2.2}$$

where α is a grey absorption coefficient and σ is the Stefan–Boltzmann constant. It can be seen that this approximation apparently reduces to the thin $(\alpha \to 0)$ and thick $(\alpha \to \infty)$ limits (Dudis 1973, thin limit; Goody 1956, thick limit) if we identify α as the Planck mean α_P in the thin limit and as the Rosseland mean α_R in the thick limit.

The complete system of equations and boundary conditions to be solved is (2.1) and (2.2). We non-dimensionalize the equations using L, ΔU, $\Delta\theta$, $\rho_0(\Delta U)^2$ and $L/\Delta U$ as the length, velocity, temperature, pressure and time scales, respectively. The radiative heat flux $\mathbf{q}$ will be scaled against the linearized Newtonian heat flux $16\sigma\alpha L T_0^3\Delta\theta$. Henceforth, all quantities will be in non-dimensional form and the variables given above will now represent non-dimensional quantities unless otherwise specified. Thus, in non-dimensional form the linearized Boussinesq and Milne–Eddington equations and boundary conditions are given by

$$\frac{d\mathbf{u}}{dt} + w\frac{\partial \mathbf{U}}{\partial z} = -\nabla p + Ri\phi\mathbf{k} + \frac{1}{Re}\nabla^2\mathbf{u}, \tag{2.3a}$$

$$\frac{d\phi}{dt} + w\frac{\partial\theta}{\partial z} = \frac{1}{P\,Re}\nabla^2\phi - \frac{16\tau}{Bo}\nabla.\mathbf{q}, \tag{2.3b}$$

$$\nabla . \mathbf{u} = 0, \tag{2.3c}$$

$$\nabla(\nabla . \mathbf{q}) - 3\tau^2\mathbf{q} - \nabla\phi = 0, \tag{2.3d}$$

$$\mathbf{u}, \phi, p, \mathbf{q} \to 0 \quad \text{as} \quad z \to \pm\infty, \tag{2.3e}$$

where $U = \tanh z$, $\partial\theta/\partial z = \operatorname{sech}^2 z$ and $\mathbf{k}$ is a unit vector in the positive-z direction. The Reynolds number Re, Prandtl number P, Richardson number Ri, Boltzmann number Bo and optical thickness τ are given by

$$Re = \frac{\Delta UL}{\nu}, \quad Ri = \frac{g(\Delta\theta/L)}{T_0(\Delta U/L)^2}, \quad P = \frac{\nu}{\kappa/\rho_0 C_p}, \quad Bo = \frac{\rho_0 c_p \Delta U}{\sigma T_0^3}, \quad \tau = \alpha L. \tag{2.4}$$

Normal modes will be employed and if f is any disturbance quantity, then $f(\mathbf{x}, t) = \hat{f}(z)\exp i(kx + ly - kct)$, where $c = c_r + ic_i$. We shall consider two-dimensional disturbances ($\hat{v} = l = 0$), and owing to the antisymmetry of the basic profiles the wave speed $c_r = 0$. Letting $\mathbf{q} = (s, 0, r)$ and $G = 16\tau/Bo$, we find that $\hat{w}$, $\hat{\phi}$, $\hat{s}$ and $\hat{r}$ satisfy the equations

$$M^2\hat{w}/k\,Re = i\{(U - ic_i)\,M\hat{w} - U''\hat{w}\} + k\,Ri\,\hat{\phi}, \tag{2.5a}$$

$$M\hat{\phi}/P\,Re = ik(U - ic_i)\,\hat{\phi} + \theta'\hat{w} + G(\hat{r}' - ik\hat{s}), \tag{2.5b}$$

$$\hat{r}' + ik\hat{s} = (3\tau^2/ik)\,\hat{s} + \hat{\phi}, \tag{2.5c}$$

$$\hat{s}' = ik\hat{r}, \tag{2.5d}$$

where $M = d^2/dz^2 - k^2$ and a prime represents d/dz. This system is to be solved subject to

$$\hat{w}, \hat{\phi}, \hat{s}, \hat{r} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{2.6}$$

It can be seen from (2.5c) that, if $|3\tau^2/ik| \ll |ik|$ or $\tau^2 \ll k^2$, then $\hat{r}' + ik\hat{s} \approx \hat{\phi}$, and we recover the optically thin equations of Dudis (1973). Thus, it appears that as $k \to 0$ the results obtained by employing the Milne–Eddington approximation may differ from those obtained by employing the optically thin approximation. This will be verified in the following sections.

The vertical component $\hat{r}$ of the heat flux may be eliminated from (2.5b, c, d) and the complete system of equations may be written as

$$M^2\hat{w}/k\,Re = i\{(U - ic_i)\,M\hat{w} - U''\hat{w}\} + k\,Ri\,\hat{\phi}, \tag{2.7a}$$

$$M\hat{\phi}/P\,Re = \{ik(U - ic_i) + G\}\,\hat{\phi} + \theta'\hat{w} + (3G\tau^2/ik)\,\hat{s}, \tag{2.7b}$$

$$M\hat{s} - 3\tau^2\hat{s} - ik\hat{\phi} = 0, \tag{2.7c}$$

which are to be solved subject to

$$\hat{w}, \hat{\phi}, \hat{s} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{2.8}$$

3. Analysis

3.1. *Optically thick limit*

For $\tau \gg 1$ we shall assume that (2.5c) may be approximated by

$$(3\tau^2/ik)\,\hat{s} + \hat{\phi} = 0. \tag{3.1}$$

This is equivalent to neglecting the first term in the Milne–Eddington equation (2.3d). Under this approximation (2.5b, d) and (3.1) may be combined to give the optically thick form of the energy equation:

$$M\hat{\phi}/P\,Re = ik(U - ic_i)\,\hat{\phi} + \theta'\hat{w} - HM\hat{\phi}, \tag{3.2}$$

where $H = 16/3\tau Bo$ represents the ratio of radiative to convective heat transport in the basic state under the optically thick approximation (see Goody 1956). The parameter H, appropriate in the optically thick limit, is related to the parameter G, appropriate in the optically thin limit, by

$$H = G/3\tau^2. \tag{3.3}$$

The optically thick form of the energy equation also may be readily derived from (2.7*b*, *c*) by expanding $\hat{s}$ as

$$\hat{s} = \hat{s}_0 + \tau^{-2}\hat{s}_2 + \tau^{-4}\hat{s}_4 + \dots .$$

Substituting this into (2.7*c*) and retaining only the highest-order terms in τ (those of order τ^2) results in the standard non-radiating energy equation. Retaining the terms of next highest order (those of order τ^0) gives (3.2).

The condition that $\tau \gg 1$ is equivalent to requiring the vertical length scale L of the shear layer to be greater than some photon mean free path $\lambda = \alpha^{-1}$. For the case of optically thick temperature disturbances the appropriate path length is the Rosseland mean free path λ_R, which is the reciprocal of the Rosseland mean absorption coefficient α_R (see Goody 1964, p. 58). Since both the vertical and horizontal scales of temperature disturbances are found to be at least as large as L, the condition that $L \gg \lambda_R$ is sufficient for (3.2) to be a valid approximation.

Now define a radiatively modified Prandtl number P_R by $1/P_R = 1/P + Re\,H$. Then the energy equation (3.2) becomes the standard non-radiating equation, though at the reduced Prandtl number P_R. The effect of reduced Prandtl number P_R on the stability of a stably stratified shear layer has recently been considered by Miller & Gage (1972). For reduced Prandtl numbers, they find increased critical Richardson numbers at finite Reynolds numbers. However, as the Reynolds number increases, the effect of reduced Prandtl number decreases, and in the inviscid limit it disappears entirely for fixed but small Prandtl numbers.

The inviscid limit of the current problem corresponds to the double limit, in Miller & Gage's investigation, $Re \to \infty$ and $P_R \to 0$ such that $1/Re\,P_R \to H$. This inviscid limit is the major interest in the current investigation, for two reasons. First, a complete destabilization (no critical Richardson number exists) was found in this limit for the optically thin model (Dudis 1973), and we wish to examine this possibility in the optically thick limit. Second, in order for (3.2) to be valid, the shear-layer depth and consequently the corresponding Reynolds number must be extremely large under typical terrestrial conditions (at sea level with 100 % relative humidity, λ_R is greater than 2 km and increases rapidly with increasing altitude and decreasing humidity).

Thus, we shall consider the inviscid limit $Re \to \infty$, in which (2.7*a*) and (3.2) may be written as

$$k\,Ri\,\hat{\phi} + i\{(U - ic_i)\,M\hat{w} - U''\hat{w}\} = 0, \tag{3.4a}$$

$$ik(U - ic_i)\,\hat{\phi} + \theta'\hat{w} - HM\hat{\phi} = 0. \tag{3.4b}$$

These equations are to be solved subject to

$$\hat{w}, \hat{\phi} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{3.5}$$

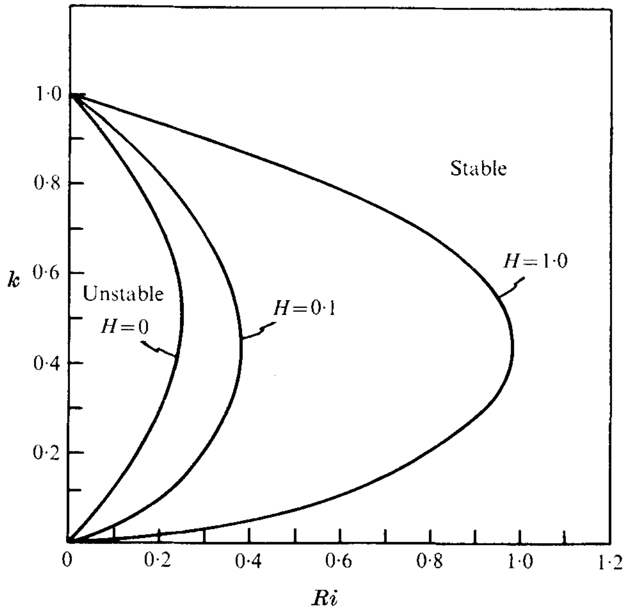

FIGURE 1. Optically thick neutral-stability boundary.

The neutral-stability boundary is the solution of the above eigenvalue problem for which $c_i = 0$. As an alternative to finding analytic solutions in the neighbourhood of the origin from which to begin numerical integration, we shall consider the non-singular system with a small but finite growth rate. This non-singular system may be solved by the same numerical procedure as was employed by Dudis (1973) in solving the viscous, optically thin problem; the details are given in the appendix. It is found that $c_i = 10^{-4}$ is a sufficiently small value of the growth rate to give an accurate approximation to the neutral-stability boundary. Decreasing c_i below 10^{-4} does not change the first three significant digits of the resulting eigenvalues. Results for $H = 0{\cdot}1$ and $1{\cdot}0$ are given in figure 1. The curve labelled $H = 0$ is the non-radiating analytic solution $Ri = k(1-k)$ (see Drazin & Howard 1966). It is to be noted that the small wavenumber instability of the optically thin model (Dudis 1973) is not present in the optically thick limit.

As in the optically thin limit, it also may be shown in the optically thick case that, for a fixed wavenumber and a sufficiently large value of the radiation parameter H, the Richardson number is a linear function of this parameter. If $Hk^2 \gg k$ or $Hk \gg 1$ then it would appear that (3.4b) could be approximated by

$$\theta'\hat{w} - HM\hat{\phi} = 0. \tag{3.6}$$

Letting $\overline{\phi} = H\hat{\phi}$ in (3.4a) and (3.6) we arrive at

$$k(Ri/H)\overline{\phi} + i(U-ic_i)M\hat{w} - iU''\hat{w} = 0, \tag{3.7a}$$

$$\theta'\hat{w} - M\overline{\phi} = 0. \tag{3.7b}$$

Thus, Ri/H is a function of k for sufficiently large Hk. This is illustrated in figure 2, where we have plotted Ri/H $vs.$ k for several values of H. The curve

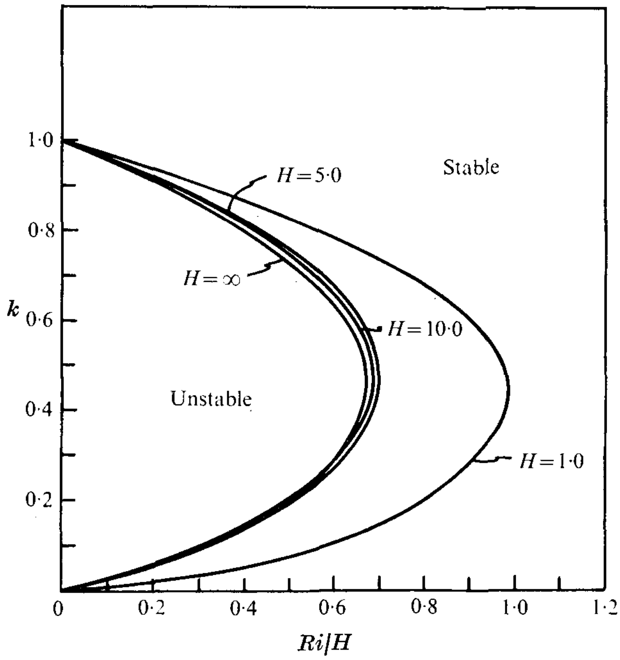

FIGURE 2. Optically thick Ri/H boundary.

marked $H = \infty$ is a solution of the limiting system (3.5) and (3.7), and for this system it is found that the critical Richardson number Ri_c (maximum of Ri with respect to k) is given by

$$Ri_c = 0{\cdot}667H. \tag{3.8}$$

For $H = 5$, Ri_c obtained from (3.8) is within 3 % of the correct value, and the approximation improves as H increases further.

3.2. *Inviscid case, general opacity*

In the limit $Re \to \infty$ the complete inviscid system may be written as

$$(U - ic_i)\, M\hat{w} - U''\hat{w} - ik\, Ri\, \hat{\phi} = 0, \tag{3.9a}$$

$$\{ik(U - ic_i) + G\}\, \hat{\phi} + \theta'\hat{w} + (3G\tau^2/ik)\, \hat{s} = 0, \tag{3.9b}$$

$$M\hat{s} - 3\tau^2\hat{s} - ik\hat{\phi} = 0, \tag{3.9c}$$

$$\hat{w}, \hat{\phi}, \hat{s} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{3.9d}$$

These equations are equivalent to a fourth-order ordinary differential equation. The neutral-stability equations ($c_i = 0$) are singular at $z = 0$, where $U = 0$, and again we shall approximate the neutral eigensolutions by keeping a small but non-zero growth rate. The numerical procedure employed (see appendix) is the same as that employed for the optically thick case.

We shall examine first the behaviour of the neutral-stability boundary for the case of small optical thickness and small wavenumber. In figure 3 we have plotted a region of the neutral-stability diagram (Ri *vs.* k) for a fixed G of 0·5 and

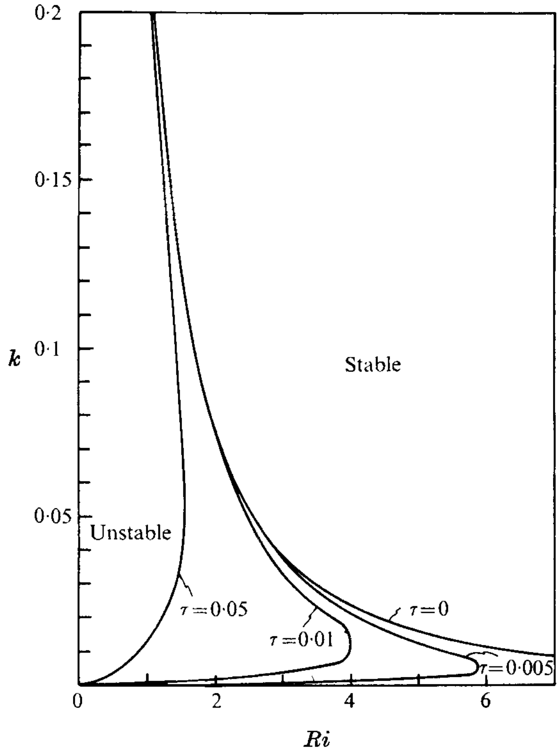

FIGURE 3. Inviscid Milne–Eddington neutral-stability boundary for $G = 0{\cdot}5$.

for several values of τ. The curve marked $\tau = 0$ is the optically thin result obtained by Dudis (1973). It can be seen, in contrast to the optically thin result, that a critical Richardson number exists for all non-zero τ. However, the critical value goes to infinity as $\tau \rightarrow 0$. This result is consistent with the observation made in § 2, that the Milne–Eddington approximation reduces to the thin approximation if $\tau^2 \ll k^2$. Thus, in figure 3, as τ is decreased, the Richardson numbers from the self-absorbing Milne–Eddington approximation follow the optically thin ($\tau = 0$) result to progressively lower values of k before diverging and returning to zero. It also should be noted that the critical Richardson number Ri_c occurs approximately at $k = \tau$. Since k has been non-dimensionalized by L, this means that the greatest destabilization takes place for disturbances whose horizontal wavelength λ_d (dimensional) is just 2π times the photon mean-free-path length $\lambda = \alpha^{-1}$. It will be shown below that this is a general feature of the small-τ case, and not merely peculiar to $G = 0{\cdot}5$.

We shall digress briefly from the case of small optical thickness and examine the behaviour of the critical Richardson number Ri_c over the entire opacity range from thin to thick. In figure 4(a) we have plotted Ri_c *vs.* τ for fixed G. Since $G = 16\tau/Bo = 16\alpha\sigma LT_0^3/\rho_0 c_p \Delta U$, keeping G fixed and varying τ corresponds to varying the depth of the shear layer while keeping the mean velocity gradient $\Delta U/L$ and fluid properties α, ρ_0, c_p and T_0 fixed. In this case $Ri_c \rightarrow \infty$ as $\tau \rightarrow 0$, the thin limit. It should be remembered that these inviscid results will

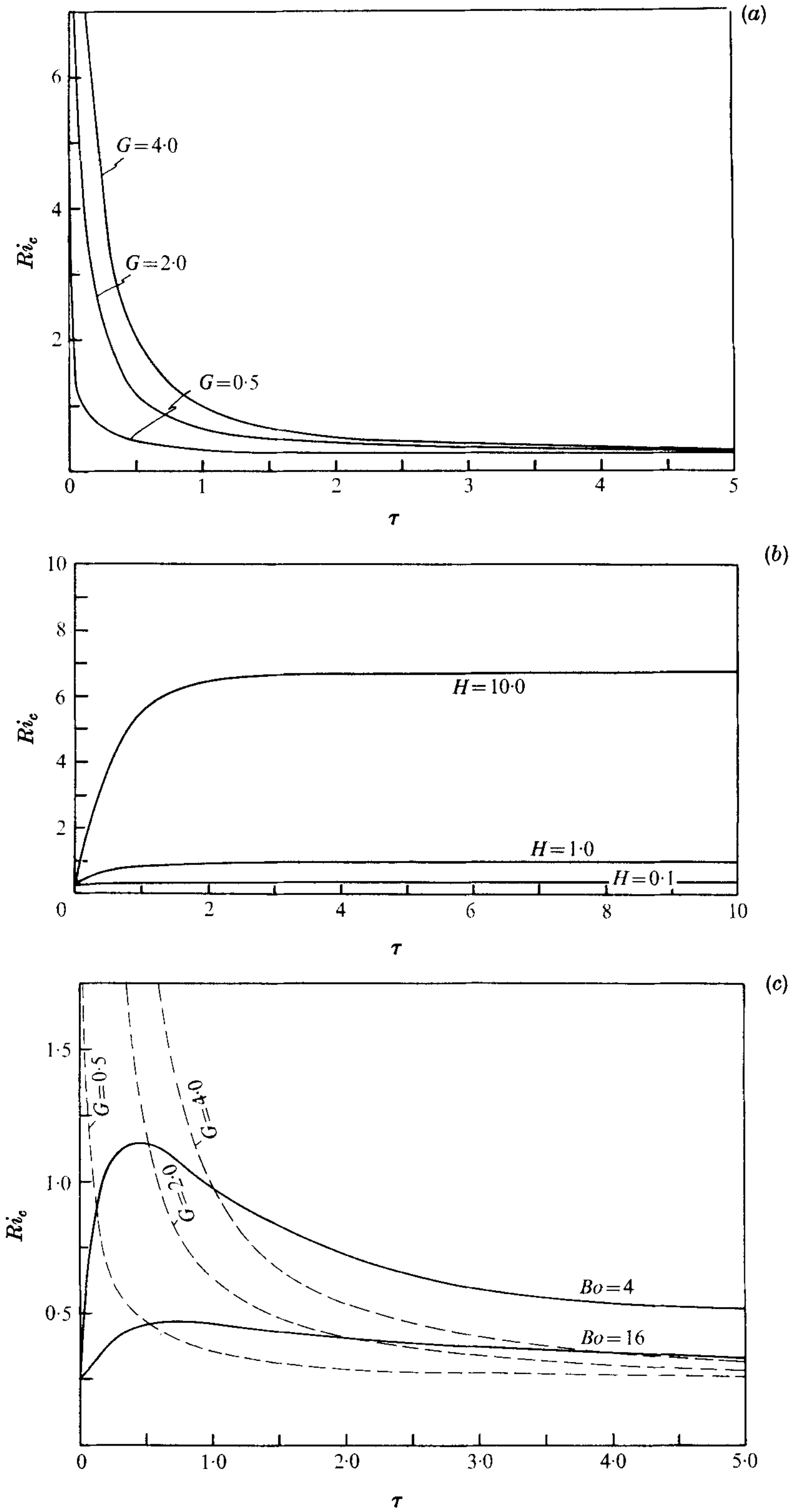

FIGURE 4. Inviscid Milne–Eddington Ri_c *vs.* τ for (*a*) fixed G, (*b*) fixed H and (*c*) fixed Bo.

be modified by viscosity as $\tau \to 0$, since in this limit Re also tends to zero for fixed ν and α. In the limit $\tau \to \infty$, the thick limit, $Ri_c \to 0{\cdot}25$. We recall that the thin and thick radiation parameters (G and H, respectively) are related by $G = 3\tau^2 H$. Thus, for fixed G, H decreases as τ increases, and in contrast to (3.8), the critical Richardson number Ri_c approaches the non-radiating value of $0{\cdot}25$, as indicated in figure 4(a).

A further view of the optically thick side of the Milne–Eddington approximation is given by figure 4(b). Here we have plotted Ri_c *vs.* τ for fixed values of the optically thick parameter H. It can be seen that the Milne–Eddington results approach the optically thick limit rapidly. For $\tau \geqslant 5$ the critical Richardson numbers have essentially reached their optically thick values given in § 2.

In figure 4(c), we have plotted Ri_c *vs.* τ for several values of the Boltzmann number Bo (the dashed lines for fixed G are included for reference). Keeping Bo fixed and varying τ corresponds to changing the depth of the shear layer, while keeping the velocity difference across the shear fixed (again for fixed α, ρ_0, c_p and T_0). The greatest destabilization occurs for τ between $0{\cdot}4$ and $0{\cdot}8$ (for sufficiently small Bo it will be seen, in connexion with figure 5(b), that the maximum critical Richardson number occurs at $\tau \approx 0{\cdot}3$). The critical Richardson number returns to its non-radiating value of $0{\cdot}25$ for $\tau \to 0$ and $\tau \to \infty$.

We shall consider the inviscid stability problem, again for the limiting case $G/k \gg 1$. This limit is of considerable importance for small τ since the critical Richardson number occurs for $k \approx \tau$, and thus the limiting results may be applicable even for relatively small G. Assume that for $G/k \gg 1$ equation (3.9b) may be written as

$$G\hat{\phi} + \theta'\hat{w} + (3G\tau^2/ik)\,\hat{s} = 0. \tag{3.10}$$

If we now make the change of variables

$$\overline{\phi} = G\hat{\phi}, \quad \bar{s} = iG\hat{s}, \quad \overline{w} = \hat{w}, \tag{3.11}$$

then (3.9a, c, d) and (3.10) may be written as

$$(U - ic_i)\,M\overline{w} - U''\overline{w} - ik(Ri/G)\,\overline{\phi} = 0, \tag{3.12a}$$

$$\overline{\phi} + \theta'\overline{w} - (3\tau^2/k)\,\bar{s} = 0, \tag{3.12b}$$

$$M\bar{s} - 3\tau^2\bar{s} + k\overline{\phi} = 0, \tag{3.12c}$$

$$\overline{w}, \overline{\phi}, \bar{s} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{3.12d}$$

In this limit Ri/G is a function of k and τ, and the critical Richardson number is a function of τ alone. That is,

$$Ri_c/G = g(\tau), \tag{3.13}$$

for $G/k \gg 1$. For $\tau \gg 1$ we know that Ri_c approaches its thick limit, where (3.8) applies for $H \gg 1/k$. Comparing (3.13) and (3.8), and since $G = 3\tau^2 H$, we see, that, as $\tau \to \infty$, $g(\tau) \to 0{\cdot}667/3\tau^2$, and this is valid only for $H = G/3\tau^2 \gg 1/k$. Thus, although (3.12) will be an accurate approximation for $G \gg k$ if $\tau < 1$, it is valid only for $G \gg 3\tau^2/k$ for large τ. An alternative approach may be employed to show that $G/k \gg 1$ is not a sufficient criterion to apply in order for (3.12) to be an accurate approximation. If we assume that for large τ (3.9c) may be approxi-

$G \backslash \tau$	1·0	0·1	0·01	0·001	0·0001
0·1	2·71	3·07	6·22	24·8	83·1
0·5	1·05	2·04	7·98	27·4	85·4
1·0	0·463	2·10	8·25	27·8	85·7
10·0	0·198	2·21	8·55	28·1	90·2
∞	0·181	2·23	8·56	28·1	90·2

TABLE 1. Critical values of Ri/G for several values of G and τ

mated by $3\tau^2\hat{s} \approx -ik\hat{\phi}$, the last term in (3.9*b*) is $\approx -G\hat{\phi}$, and thus we cannot neglect $ik(U-ic_i)$ with respect to G as was done in arriving at (3.10).

In table 1 we have listed values of the critical Richardson number divided by G for various values of G and τ. The last line for $G = \infty$ represents the results for the limiting system (3.12). For a given value of G, it can be seen that the approximation improves appreciably as $\tau \to 0$. For $G \geqslant 0{\cdot}5$ and $\tau < 0{\cdot}1$ the large-G/k limit is within 10% of the actual Ri_c/G for given G and τ.

It was mentioned above for the case of $G = 0{\cdot}5$ that as $\tau \to 0$ the critical Richardson number occurs at $k \approx \tau$. This is also true for the large-G/k limit. For a given value of τ the critical value of Ri_c/G is found to lie at $k \approx \tau$ and to be approximately 0·62 times the transparent value of Ri/G (Dudis 1973) at the same k. Since Ri/G varies as $k^{-\frac{1}{2}}$ for $k \to 0$ in the transparent large-G/k limit, it would be expected that $Ri_c/G \sim \tau^{-\frac{1}{2}}$ as $\tau \to 0$. This is indicated in figure 5(*b*), where the dashed line represents, empirically,

$$Ri_c \approx 0{\cdot}9G\tau^{-\frac{1}{2}}, \tag{3.14}$$

for $\tau \to 0$. For any fixed G it is possible to find a small enough τ such that (3.14) is an accurate approximation to the actual critical value.

Since Ri_c is a linear function of G (for fixed τ) in the large-G limit, we also have that Ri_c is a linear function of Bo^{-1}. In figure 5(*b*) we have plotted $Ri_c/16Bo^{-1}$ *vs.* τ for this large-G limit. From this graph it can be seen that for sufficiently small Bo (large G) the maximum destabilizing effect for fixed Boltzmann number occurs at an opacity of about 0·3. It appears that $Ri_c \to 0$ as $\tau \to 0$, but it must be remembered that these results are a valid approximation for arbitrary Bo only for G sufficiently large; in the limit $\tau \to 0$, Bo fixed, G also tends to zero, and the approximation becomes invalid. The smaller the value of Bo, the smaller the value of τ may be for which figure 5(*b*) gives an accurate prediction of the critical Richardson number.

After digesting all the above results it becomes evident that the greatest destabilization in this inviscid Milne–Eddington model occurs when considering shear layers of fixed mean shear $\Delta U/L$ and letting the scale L of the shear layer (and consequently the optical thickness τ) approach zero. For this case (3.14) eventually becomes valid, and the critical Richardson number approaches infinity as the shear depth approaches zero. However, as the shear layer becomes thinner, viscosity should play an increasingly important role as a stabilizing mechanism. In the following subsection this viscous effect will be considered.

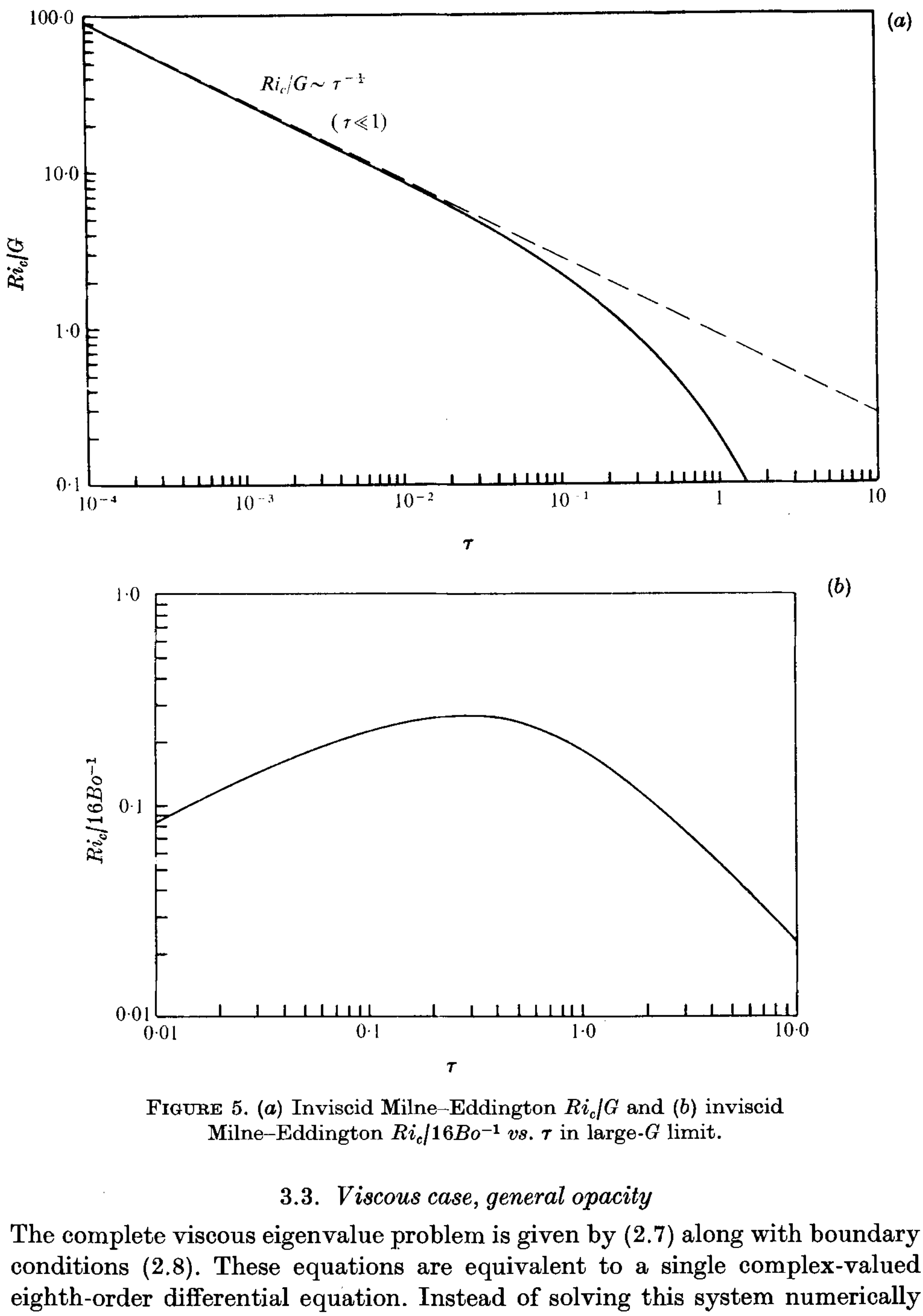

FIGURE 5. (*a*) Inviscid Milne–Eddington Ri_c/G and (*b*) inviscid Milne–Eddington $Ri_c/16Bo^{-1}$ *vs.* τ in large-G limit.

3.3. *Viscous case, general opacity*

The complete viscous eigenvalue problem is given by (2.7) along with boundary conditions (2.8). These equations are equivalent to a single complex-valued eighth-order differential equation. Instead of solving this system numerically we shall look again at the limiting case of large G/k. One reason for treating the approximate system is that, for the interesting case of fixed G and $\tau \to 0$, the critical Richardson number will be located at small τ and k, where moderate values of G are sufficiently large for the approximation to be accurate. Second,

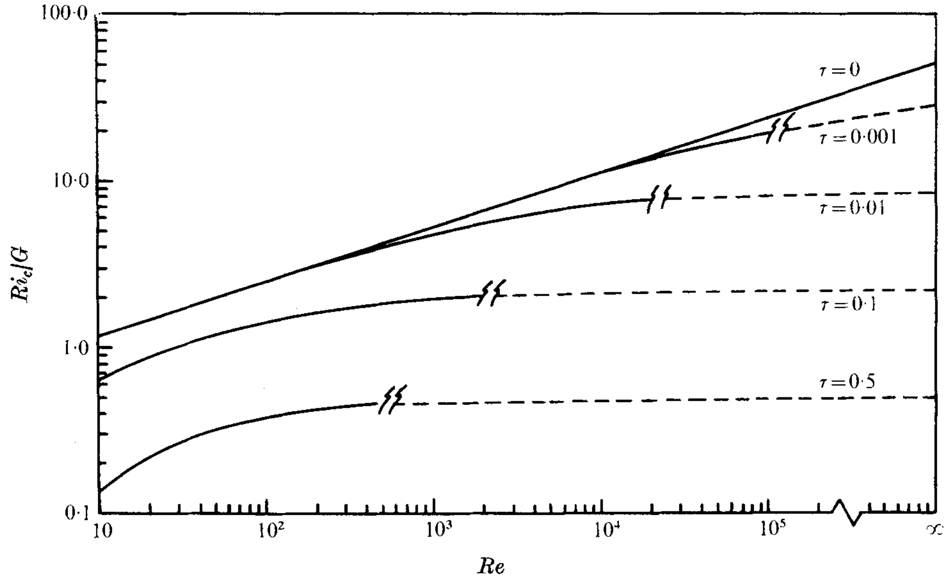

FIGURE 6. Viscous Milne–Eddington Ri_c/G *vs.* Re in large-G limit.

we shall again be able to show that Ri_c is a linear function of G for fixed τ and Re, and thus, provided that G is large enough, we shall have avoided the necessity of having to carry out a separate computation for each distinct value of G. Third, the approximation will reduce the number of linearly independent decaying solutions from four to three, and it also will eliminate the numerically troublesome solutions which (for $z < 0$) grow as $\exp(-GP\,Re^{\frac{1}{2}}z)$ (see Dudis 1973).

We shall employ the transformation of variables (3.11) and assume that $G/k \gg 1$ and $GP\,Re \gg 1$. Under these conditions, and assuming neutral stability ($c_i = 0$), the system (2.7) and (2.8) may be approximated by

$$M^2\overline{w} - k\,Re\{i(UM\overline{w} - U''\overline{w}) - k(Ri/G)\,\overline{\phi}\} = 0, \tag{3.15a}$$

$$\theta'\overline{w} - (3\tau^2/k)\,\overline{s} + \overline{\phi} = 0, \tag{3.15b}$$

$$M\overline{s} - 3\tau^2\overline{s} + k\overline{\phi} = 0, \tag{3.15c}$$

$$\overline{s}, \overline{\phi}, \overline{w} \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{3.15d}$$

Once again Ri/G becomes the new eigenvalue as a function of k, τ and Re; and for fixed Re and τ, the critical Richardson number Ri_c is a linear function of G.

The method of solution of the eigenvalue problem (3.15) is similar to that employed in treating the optically thin viscous problem (Dudis 1973). Complete details are given in the appendix. Results from the numerical computations are given in figure 6, where the straight line labelled $\tau = 0$ is the optically thin result $Ri_c/G = 0{\cdot}53Re^{\frac{1}{4}}$ from Dudis (1973). The dashed lines represent extrapolation of the viscous results up to the inviscid limit given along the right-hand ordinate. It is possible to carry out the viscous computations only up to some finite value (depending on τ) of the Reynolds number; beyond this value, growing exponential solutions lead to numerical instabilities. The most dangerous solution

$$\sim \exp[-(k\,Re)^{\frac{1}{2}}z]$$

(z is negative over the range of integration, see appendix), and it is generally possible to obtain solutions up to $k_c Re \sim 200$ (the critical wavenumber k_c is the value of the wavenumber which yields Ri_c). For fixed τ ($\leqslant 0{\cdot}1$), k_c approaches τ as $Re \to \infty$, and thus viscous results may be obtained for increasing values of Re with decreasing τ.

With increasing Reynolds number, the values of the critical wavenumbers for figure 6 vary from 0·06 to 0 along the straight line $\tau = 0$, from 0·06 to 0·001 along $\tau = 0{\cdot}001$, from 0·06 to 0·01 along $\tau = 0{\cdot}01$, from 0·2 to 0·1 along $\tau = 0{\cdot}1$ and from 0·4 to 0·3 along $\tau = 0{\cdot}5$.

From figure 6 it may be seen that the viscous (finite Reynolds number) Milne–Eddington results become the optically thin ones as $\tau \to 0$. For small enough τ ($\leqslant 0{\cdot}01$), say) and at low Reynolds number, the Milne–Eddington approximation yields critical Richardson numbers practically equal to the optically thin values. However, as the Reynolds number increases, eventually the Milne–Eddington curves branch off and approach their inviscid limit, whereas the thin approximation has no inviscid limit, and $Ri_c/G \to \infty$ as $Re^{\frac{1}{3}}$.

Also, one can infer from figure 6 that a transverse dimension L exists for which thermal radiation will have the largest destabilizing effect (i.e. the largest critical Richardson number) on a shear layer of given fluid properties α, ρ_0, T_0, ν and c_p and mean velocity gradient $\Delta U/L$. The argument is as follows. Under these conditions G is fixed, $\tau \sim L$ and $Re \sim L^2$. Assume that a given value of L corresponds to some point (τ, Re) in the lower right-hand corner of figure 6. As L is decreased, the locus of (τ, Re) moves up and to the left, since both τ and Re are decreasing. Eventually, however, we must reach a limiting value of L where any further decrease in L leads to a lowering of the point (τ, Re). This limiting value of L yields the largest critical Richardson number for the specified fluid properties and mean velocity gradient. Further, this condition will correspond to a small value of τ, as can be inferred from figure 6. Specific cases will be examined in the following section.

4. Discussion

The results of § 3 will now be applied to the examples considered previously in the optically thin limit (Dudis 1973). Before proceeding further it will be necessary to relate the grey absorption coefficient α to a real, non-grey radiating atmosphere. This will be accomplished by employing a result of Spiegel (1957) which relates the radiative decay rate N of a spherically symmetric temperature perturbation in a grey gas to a grey absorption coefficient and the length scale of the temperature perturbation. Adapting Spiegel's result to the present problem gives (see Dudis & Traugott (1974) for a more detailed discussion of this determination of an effective absorption coefficient)

$$N = \frac{16\sigma T_0^3}{\rho_0 c_p L}\tau\left(1 - \frac{\tau}{\pi}\cot^{-1}\frac{\tau}{\pi}\right). \tag{4.1}$$

For a pure CO_2 atmosphere, Goody & Belton (1967) have computed the non-grey radiative decay rate N as a function of the disturbance length scale. They also give corresponding results for the earth's atmosphere with a water vapour content similar to that which will be considered here. Thus, given the atmospheric conditions and a shear-layer length scale, we can determine N from the results of Goody & Belton; τ (and consequently an effective absorption coefficient) can be calculated by solving the transcendental equation (4.1) with this known value of N. Under most conditions which we shall consider, τ will be small and the second term in brackets in (4.1) may be neglected. Thus, the stability parameter G is approximately related to N by

$$G \approx N/S,$$

where $S = \Delta U/L$ is the mean shear rate across the shear layer.

The examples to be considered are the upper atmosphere of Venus (at 100 km) and the earth's lower troposphere and lower stratosphere. We shall consider first the 100 km level above Venus, where it is assumed that the atmosphere is entirely CO_2. As in Dudis (1973) we shall take typical values of the temperature T_0, density ρ_0 and mean shear S to be given by

$$T_0 = 150\,^\circ\mathrm{K}, \quad \rho_0 = 7{\cdot}3 \times 10^{-8}\,\mathrm{g\,cm^{-3}}, \quad S = 10^{-2}\,\mathrm{s^{-1}}.$$

This large shear is chosen to be representative of the four-day Venus circulation (Gold & Soter, 1972). For the 400 m shear layer ($L = 200$ m) considered in Dudis (1973), $Re = 3700$. We find from the procedure outlined above that $\tau = 0{\cdot}001$ and $G = 0{\cdot}025$. It should be noted that this value of τ (and therefore G as well) is much smaller than that employed previously in the optically thin analysis. This is due to the fact that the choice of the Planck mean in the previous analysis was, for the length scale considered, an overestimation of the CO_2 absorption coefficient. From figure 2 of Goody & Belton it can be determined that only for shear layers more than three orders of magnitude thinner than that considered here will radiative transfer be governed by the thin approximation with the Planck mean as the effective absorption coefficient. In the present case, the viscous large-G/k results (the critical wavenumber $k_c \approx \tau$, giving $G/k \approx 25$) from figure 5(a) give $Ri_c/G \approx 8$, corresponding to $Ri_c \approx 0{\cdot}3$. Since the inviscid large-G/k limit (last line in table 1) yields values for Ri_c/G generally greater than those obtained for finite values of G, one would expect the viscous approximate value of 0·3 to be an upper bound to the exact value. Neither increasing nor decreasing the length scale L will lead to any appreciable increase in Ri_c. For increased L, a decrease in G will reduce Ri_c; viscous effects will limit the growth of Ri_c as L is decreased. Thus, for this altitude, radiative transfer will have little effect on the stability characteristics. A more complete analysis of possible destabilization in the Venusian atmosphere is given by Dudis & Traugott (1974).

Next, we shall consider the earth's lower troposphere. As before, we shall take

$$T_0 = 300\,^\circ\mathrm{K}, \quad \rho_0 = 1{\cdot}3 \times 10^{-3}\,\mathrm{g\,cm^{-3}}, \quad S = 2 \times 10^{-3}\,\mathrm{s^{-1}}.$$

With 300 p.p.m. CO_2 (by volume) and 3 mbar of water vapour (approximately 10 % relative humidity) we find, for the 20 m shear layer ($L = 10$ m) considered previously, $\tau = 0{\cdot}11$ and $G = 0{\cdot}13$. The corresponding Reynolds number $Re = 15000$. It can be seen from figure 6 that for this relatively large value of

Re the shear layer is practically inviscid. In table 1, we note that, for $\tau = 0{\cdot}1$ and $G = 0{\cdot}1$, $Ri_c/G = 3{\cdot}07$, giving $Ri_c \approx 0{\cdot}3$. An exact inviscid calculation for $\tau = 0{\cdot}11$ and $G = 0{\cdot}13$ yields $Ri_c = 0{\cdot}33$, which is considerably less than the value of 4·2 determined from the optically thin analysis. The value of L which gives the maximum critical Richardson number is approximately one metre. For this shear layer, $\tau = 0{\cdot}052$, $G = 0{\cdot}062$, $Re = 150$ and $Ri_c \approx 1{\cdot}1$. The large-G viscous results are assumed valid since, in the inviscid case, Ri_c/G for $G = 0{\cdot}5$ and $\tau = 0{\cdot}1$ is within 9 % of the large-G/k limit; see table 1.

Qualitatively, this result for the earth's lower troposphere is similar to that obtained by Goody (1964, p. 369) in his application of Townsend's (1958) results. For the same physical parameters (mean shear, density, temperature and water vapour partial pressure), Goody finds critical Richardson numbers doubled over their non-radiating values for turbulent eddies smaller than a few metres in diameter. It should be noted, however, that Townsend's non-radiating critical Richardson number is $\frac{1}{12}$ and not the $\frac{1}{4}$ of the present analysis. Also, in the present laminar analysis viscous effects drive critical Richardson numbers back to zero as the shear-layer depth (and hence the Reynolds number) approaches zero, whereas Townsend obtains a maximum increase as the turbulent eddy size approaches zero.

The final example to be considered is the 20 km level in the earth's atmosphere, where we assume

$$T_0 = 220\,^\circ\mathrm{K}, \quad \rho_0 = 8{\cdot}9 \times 10^{-5}\,\mathrm{g\,cm^{-3}}, \quad S = 2 \times 10^{-3}\,\mathrm{s^{-1}},$$

with 300 p.p.m. CO_2 (by volume) and no water present. Under these conditions and for a 240 m deep shear ($L = 120$ m), $\tau = 7 \times 10^{-3}$, $G = 3{\cdot}8 \times 10^{-3}$ and $Re = 180\,000$. An inviscid calculation for these values of G and τ yields a critical Richardson number of about 0·25. The inclusion of viscosity could only lower this value: thus, for a shear layer of this thickness radiative destabilization is negligible. At this altitude, the same result is found for all shear-layer thicknesses. Again, this is qualitatively similar to the conclusion reached by Townsend (1958), who was unable to explain the existence of observed stratospheric turbulence on the basis of radiative destabilization.

The author thanks S. C. Traugott for many helpful discussions during the investigation. He also thanks R. D. Cess for pointing out an overestimation of non-grey atmospheric opacities in an earlier draft of this paper.

Appendix. Solution of the eigenvalue problems

Inviscid optically thick system

Owing to the antisymmetry of the basic profiles U and θ, the system (3.4) and (3.5) possesses a solution $(\hat{w}, \hat{\phi})$ such that $(\hat{w}(z), \hat{\phi}(z)) = (\hat{w}^*(-z), \hat{\phi}^*(-z))$ (where $\hat{w}^*$ represents the complex conjugate of $\hat{w}$). This leads to the boundary conditions

$$\mathrm{Re}\,(\hat{w}', \hat{\phi}') = \mathrm{Im}\,(\hat{w}, \hat{\phi}) = 0 \quad \text{at} \quad z = 0. \tag{A 1}$$

As $z \to -\infty$, $U \to -1$ and U'', $\theta' \to 0$. Under these conditions

$$\hat{w}_i \sim e^{\lambda_i z}, \quad \hat{\phi}_i \sim \delta_{2i} e^{\lambda_i z} \quad (i = 1, 2), \tag{A 2a}$$

where δ_{ij} is the Kronecker delta and

$$\lambda_1 = k, \quad \lambda_2 = k[1+(kH)^{-1}(c_i - i)]^{\frac{1}{2}}. \qquad \text{(A 2}b\text{)}$$

Solutions (A 2) are used as starting values to begin numerical integration from some large negative $z = \bar{z}$ to the origin. These two linearly independent solutions $(\hat{w}_i, \hat{\phi}_i)$ $(i = 1, 2)$ are related to the actual eigenfunctions (which satisfy both (A 1) and (A 2)) by

$$\left.\begin{aligned} \hat{w} &= (1+iA)\,\hat{w}_1 + (B+iC)\,\hat{w}_2, \\ \hat{\phi} &= (1+iA)\,\hat{\phi}_1 + (B+iC)\,\hat{\phi}_2, \end{aligned}\right\} \qquad \text{(A 3)}$$

where A, B and C are real constants. Application of boundary conditions (A 1) to (A 3) yields four algebraic equations for the three unknown constants A, B and C. As in Dudis (1973), we solve two different sets of three equations each, and in general find different values of A, B and C. If δA, δB and δC represent the differences between these, we vary Ri (with k, τ, Bo and Re fixed) until all of the δ's change sign between two values of Ri. Newton's method is then used to find the correct eigenvalue, where all of the differences are zero (meaning that both sets of equations yield the same starting values for $\hat{w}$ and $\hat{\phi}$).

As was pointed out by Dudis (1973), the magnitude of $\bar{z}$ which is necessary for (A 2) to be a valid approximate solution depends on the wavenumber k ($e^{2\bar{z}} \ll k^2$). For small k, this implies that the equations must be integrated numerically for a considerable distance from the origin. In order to avoid the necessity of integrating the equations over large distances we shall again employ the method of Gage (1972). In this method the basic profiles are approximated by -1 for $z \leqslant z_1$ (generally z_1 is taken as -3). For $z = z_1$ solutions (A 2) are valid. The values of $\hat{w}_1$ and $\hat{\phi}_i$ on either side of the singularity in the basic profiles at $z = z_1$ are found by employing the jump conditions determined from (3.4). The method of solution then proceeds as described above. Complete details for determining the jump conditions are given in Gage (1972) and Dudis (1973). For the present problem these conditions are found to be

$$\left.\begin{aligned} [V\hat{w}_i' - V'\hat{w}_i] &= 0, \quad [\hat{w}_i/V] = 0, \\ [\tfrac{1}{2}V\hat{w}_i - H\hat{\phi}_i] &= 0, \quad\quad [\hat{\phi}_i] = 0, \end{aligned}\right\} \qquad \text{(A 4)}$$

where $V = U - ic_i$ and $[\hat{\phi}]$ represents the jump in $\hat{\phi}$ across the singularity at $z = z_i$. If z_1 is decreased beyond -3 no change is found in the resulting eigenvalues to three significant digits.

Inviscid Milne–Eddington system

Equation (3.9b) may be solved for $\hat{\phi}$ in terms of $\hat{w}$ and $\hat{s}$. Substituting the result into (3.9a, c) and letting $\bar{s} = iG\hat{s}$ yields the following two equations in two unknowns:

$$VM\hat{w} - V''\hat{w} + \frac{ik\,Ri}{G+ikV}\left(\theta'\hat{w} - \frac{3\tau^2\bar{s}}{k}\right) = 0,$$

$$M\bar{s} - 3\tau^2\bar{s} - \frac{Gk}{G+ikV}\left(\theta'\hat{w} - \frac{3\tau^2\bar{s}}{k}\right) = 0. \qquad \text{(A 5)}$$

These equations are of the same order (equivalent to a single fourth-order equation) as the inviscid optically thick equations. The method of solution is the same as that described above, except that we now have decaying exponential solutions different from (A 2) and jump conditions different from (A 4). Boundary conditions (A 1) apply with $\bar{s}$ replacing $\bar{\phi}$.

In the inviscid Milne–Eddington system (A 5) the two decaying exponential solutions for $U = -1$ and $U'' = \theta' = 0$ are given by

$$\hat{w}_i \sim e^{\lambda_i z}, \quad \bar{s}_i \sim \delta_{2i} e^{\lambda_i z} \quad (i = 1, 2), \tag{A 6a}$$

where
$$\lambda_1 = k, \quad \lambda_2 = \left\{k^2 + 3\tau^2 - \frac{3G\tau^2(G+kc_i+ik)}{(G+kc_i)^2+k^2}\right\}^{\frac{1}{2}}. \tag{A 6b}$$

The jump conditions across the singularity at $z = z_1$ are given by

$$\left.\begin{aligned} \left[V\hat{w}_i' - V'\hat{w}_i + Ri\frac{\hat{w}_i}{V}\left\{V + \frac{iG}{k}\ln\left(V - \frac{iG}{k}\right)\right\}\right] = 0, \quad \left[\frac{\hat{w}_i}{V}\right] = 0, \\ \left[\bar{s}_i' + iG\frac{\hat{w}_i}{V}\left\{V + \frac{iG}{k}\ln\left(V - \frac{iG}{k}\right)\right\}\right] = 0, \quad [\bar{s}_i] = 0. \end{aligned}\right\} \tag{A 7}$$

Again $z_1 = -3$ and $c_i = 10^{-4}$ are employed. A decrease in either of these values causes no change, to three significant figures, in the resulting eigenvalues.

Viscous Milne–Eddington system, large-G/k limit

If (3.15*b*) is solved for $\bar{\phi}$ in terms of $\bar{w}$ and $\bar{s}$, this equation may be substituted into (3.15*a*, *c*) yielding two equations in the two unknowns, $\bar{w}$ and $\bar{s}$:

$$\left.\begin{aligned} \frac{M^2\bar{w}}{k\,Re} - i(UM\bar{w} - U''\bar{w}) + \frac{Ri}{G}(k\theta'\bar{w} - 3\tau^2\bar{s}) = 0, \\ M\bar{s} - k\theta'\bar{w} = 0. \end{aligned}\right\} \tag{A 8}$$

These equations are equivalent to a complex-valued sixth-order ordinary differential equation. Again, owing to the antisymmetry of the basic profiles U and θ, complex eigenfunctions $\bar{w}$ and $\bar{s}$ with $\bar{w}(z) = \bar{w}^*(-z)$ and $\bar{s}(z) = \bar{s}^*(-z)$ exist. This determines the boundary conditions

$$\mathrm{Re}\,(\bar{w}', \bar{w}''', \bar{s}') = \mathrm{Im}\,(\bar{w}, \bar{w}'', \bar{s}) = 0 \quad \text{at} \quad z = 0. \tag{A 9}$$

We again approximate the basic profiles by $U = \theta \equiv -1$ for $z \leqslant z_1$, and we have the following three decaying solutions:

$$\bar{s}_i \sim \delta_{1i}\, e^{\lambda_i z}, \quad \bar{w}_i \sim (z\delta_{1i} + \delta_{2i} + \delta_{3i})\, e^{\lambda_i z} \quad (i = 1, 2, 3), \tag{A 10a}$$

where
$$\lambda_1 = k, \quad \lambda_2 = k, \quad \lambda_3 = k(1 - i\,Re/k)^{\frac{1}{2}}. \tag{A 10b}$$

Jump conditions across the singularity are needed in order to continue the independent solutions (A 10) across z_1, whence numerical integration of (A 8) may proceed towards the origin $z = 0$. These jump conditions are found from (A 8) to be given by

$$\left.\begin{aligned} [\bar{w}_i'''' - ik\,Re\,(U\bar{w}_i' - U'\bar{w}_i) + k^2\,Re\,(Ri/G)\,\theta\bar{w}_i] = 0, \\ [\bar{w}_i'' + ik\,Re\,U\bar{w}_i] = 0, \quad [\bar{w}_i'] = 0, \quad [\bar{w}_i] = 0, \\ [\bar{s}_i - k\theta\bar{w}_i] = 0, \quad [\bar{s}_i] = 0. \end{aligned}\right\} \tag{A 11}$$

The actual eigenfunctions $\overline{w}$ and $\overline{s}$ may be written as a linear combination of the three independent decaying solutions. Therefore,

$$\left.\begin{aligned} \overline{w} &= (1+iA)\,\overline{w}_1 + (B+iC)\,\overline{w}_2 + (D+iE)\,\overline{w}_3, \\ \overline{s} &= (1+iA)\,\overline{s}_1 + (B+iC)\,\overline{s}_2 + (D+iE)\,\overline{s}_3, \end{aligned}\right\} \quad \text{(A 12)}$$

where A, B, C, D and E represent real constants. Applying boundary conditions (A 9) to (A 12) gives six equations for the five unknown constants. We solve two different sets of five equations each and, in general, calculate two different values for each of the constants. If δA, δB, δC, δD and δE represent these differences, we vary Ri (with k, τ, Bo and Re fixed) so as to decrease the magnitude of all the δ's, and eventually, between two successive values of Ri, all of the δ's will change sign. Again Newton's method is used to find the eigenvalue Ri, where all the δ's are zero.

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